ON A SPHERICAL SHOCK WAVE IN A VISCO-ELASTO-PLASTIC HALF-SPACE

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A half-space filled with a visco-elasto-plastic material is considered, and it is supposed that a pressure impulse is applied to a sphere situated at a certain depth from the free surface. Relations are obtained describing the plastic deformation of the medium, as well as the propagating zone between the compression and relaxation waves.

The reflection of a stress wave at the free surface is considered, and an analytical investigation of the state of the medium behind a reflected irrotational wave is carried out.

Expressions are obtained describing the change in intensity of a reflected irrotational wave, and it is shown that in a certain region the reflected irrotational wave will be a plastic stress wave.

The problem of the propagation of a spherical wave in an elastic prestressed half-space and its reflection from a free surface was studied by several authors [1, 2].

Here we consider an initially unstressed half-space or visco-elasto-plastic material. On a sphere Σ_0 of radius R_0 with center at the point O at a depth h from the free surface, a pressure $P_0 > k\sqrt{3}$ acts during a short time interval $[0, t_0]$, and for $t > t_0$ the pressure on the sphere Σ_0 vanishes. At a given instant $t > t_0$ a portion of the material, bounded between wave surfaces Σ_1 and Σ_2, (irrotational stress and relaxation waves propagating with velocity $c_0 = [(\lambda + 2\mu)/\rho]^{1/2}$) will be in the plastic state, whereas the material between the surfaces Σ_2 and Σ_0 will be in the elastic state.

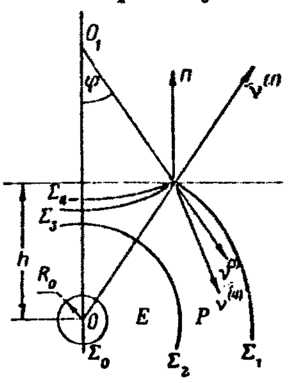

Fig. 1

At the time $t = h/c_0$ the wave surface Σ_1 reaches the free surface and for $t > h/c_0$ there are two reflected waves Σ_3 and Σ_4 in the half-space; these are the irrotational and equivoluminal waves, propagating with velocities c_0 and $c_1 = (\mu/\rho)^{1/2}$ respectively (Fig. 1, where P denotes that the medium is deformed plastically, and E denotes elastic deformation).

In this note relations are obtained describing plastic deformation of the material between the surfaces Σ_1 and Σ_2 and the behavior of the material on the reflected irrotational wave Σ_3.

1. We consider a visco-elasto-plastic material. The rheological equation of the material has the form [3]

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu (e_{ij} - e_{ij}^p) \tag{1.1}$$

$$\varepsilon_{ij}^p = \frac{\psi}{1+\eta\psi} s_{ij}', \quad (s_{ij} - \eta\varepsilon_{ij}^p)(s_{ij} - \eta\varepsilon_{ij}^p) = 2k^2$$

$$s_{ij} = \sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}, \qquad \varepsilon_{ij}^p = \partial e_{ij}^p / \partial t$$

Here λ, μ are Lamé parameters, η is the coefficient of viscosity, k is the plastic limit, and ψ a positive coefficient. The stress σ_{ij} and velocity of displacement v_i satisfy the equation of motion

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \rho \frac{\partial v_i}{\partial t} = 0 \tag{1.2}$$

Eliminating the quantities ψ, ε_{ij}^{p} from the system (1.1), (1.2) and using spherical coordinates with center at O under the assumptions

$$v_\varphi = v_\theta = 0, \quad \sigma_{r\varphi} = \sigma_{\varphi\theta} = \sigma_{\varphi\theta} = 0,$$

we obtain a system of three differential equations, describing the plastic deformation of the material between the surfaces Σ_1 and Σ_2

$$\frac{\partial \sigma_1}{\partial r} + \frac{2}{r}(\sigma_1 - \sigma_2) = \alpha \frac{\partial v}{\partial t}$$

$$\frac{\partial \sigma_1}{\partial t} = \alpha \frac{\partial v}{\partial r} + 2\beta \frac{v}{r} - \frac{4}{3}\gamma(\sigma_1 - \sigma_2 + \sqrt{3}) \tag{1.3}$$

$$\frac{\partial \sigma_2}{\partial t} = \beta \frac{\partial v}{\partial r} + (\alpha + \beta)\frac{v}{r} + \frac{2}{3}\gamma(\sigma_1 - \sigma_2 + \sqrt{3})$$

Here

$$\sigma_1 = \frac{\sigma_{rr}}{k}, \quad \sigma_2 = \frac{\sigma_{\theta\theta}}{k}, \quad v = \frac{v_r}{c_0}, \quad r = \frac{r}{h}, \quad t = \frac{t}{h}c_\cdot, \quad \alpha = \frac{\lambda + 2\mu}{k}, \quad \beta = \frac{\lambda}{k}, \quad \gamma = \frac{\mu h}{c_0 \eta}$$

are dimensionless quantities.

The solution in the zone of plasticity is represented in the form of a ray expansion

$$f_i = -\omega_i - \varepsilon\varphi_i - \tfrac{1}{2}\varepsilon^2 g_i - \dots \tag{1.4}$$

$$f_1 = \sigma_1, \quad f_2 = \sigma_2, \quad f_3 = v, \quad \varepsilon = t - r$$

The quantities

$$\omega_i = [f_i], \qquad \varphi_i = \left[\frac{\partial f_i}{\partial r}\right], \qquad g_i = \left[\frac{\partial^2 f_i}{\partial^2 r}\right]$$

are defined on the surface Σ_1.

Limiting consideration to quantities of first order in ε in (1.4) and using the condition of dynamic and kinematic compatibility [5], we obtain values for ω_3 and ϕ_3 as follows

$$\omega_3 = \frac{C_1}{r}\exp[-g(\alpha - \beta)r] - \frac{\sqrt{3}}{\alpha - \beta}\left[1 - \frac{1}{g(\alpha - \beta)r}\right]$$

$$\varphi_3 = \left[C_2 - 2\frac{C_1}{r} - C_1\frac{3}{2}q^2(\alpha^2 - \beta^2)r\right]\frac{1}{r}\exp[-g(\alpha - \beta)r] - \tag{1.5}$$

$$-\frac{\sqrt{3}}{(\alpha - \beta)^2 r}\left(4\alpha - \beta + \frac{2}{gr}\right)$$

The constants C_1 and C_2 are determined from the conditions

$$r = r_0, \qquad P_0 = -\alpha\omega_3, \qquad \varphi_1 = -\alpha\frac{\partial \omega_3}{\partial r} \tag{1.6}$$

One may in the same manner determine ω_1, ω_2, ϕ_1 and ϕ_2. The stress and velocity in the zone of elasticity between the surfaces Σ_0 and Σ_2 will not be calculated; we note, however, that they are quantities of order ε.

2. We consider the change in the intensity $\omega_3^{(1)}$ of the reflected irrotational wave Σ_3 in the process of its propagation in the direction $\nu^{(3)}$ from the point O_1. The jumps in σ_{ij} and v_i across each wave surface in the neighborhood of the reflection point M must satisfy a condition of dynamic compatibility [3]

$$-c[\sigma_{ij}] = \beta[v_k]\nu_k\delta_{ij} + \frac{\alpha - \beta}{2}([v_i]\nu_j + [v_j]\nu_i) \tag{2.1}$$

Here

$$c = c_0 = 1 \text{ for } \Sigma_2 \text{ and } \Sigma_3, \qquad c = c_1 = [(\alpha - \beta)/2\alpha]^{1/2} \text{ for } \Sigma_4$$

On the free surface the following condition should be satisfied:

$$\sigma_{ij}^{(4)} n_j = 0 \tag{2.2}$$

The conditions that Σ_3 and Σ_4 are irrotational and equivoluminal wave surfaces take the form

$$[v_i^{(3)}]\, \nu_i^{(3)} = \omega_3^{(1)}, \qquad [v_i^{(4)}]\, \nu_i^{(4)} = 0 \tag{2.3}$$

Setting

$$n_i = \{\cos\varphi,\ -\sin\varphi,\ 0\}, \qquad \nu_i^{(1)} = \{1,\ 0,\ 0\}$$

$$\nu_i^{(3)} = \{-\cos 2\varphi,\ \sin 2\varphi,\ 0\}, \qquad \nu_i^{(4)} = \{-\cos(\varphi+\psi),\ \sin(\varphi+\psi),\ 0\} \tag{2.4}$$

(where ϕ is the angle between the free surface and the wave Σ_3, and ψ is the angle between the free surface and the wave Σ_4), we solve the system of equations (2.1) to (2.3) for $\omega_3^{(1)}$ at the point M under the conditions that

$$\frac{c_0}{c_1} = \left(\frac{2\alpha}{\alpha-\beta}\right)^{1/2} = \frac{\sin\varphi}{\sin\psi} \tag{2.5}$$

$$\omega_3^{(1)} = -\zeta\omega_3, \qquad \zeta = \frac{\operatorname{tg}\varphi - \operatorname{tg}\psi\operatorname{tg}^2 2\psi}{\operatorname{tg}\varphi + \operatorname{tg}\psi\operatorname{tg}^2 2\psi} \tag{2.6}$$

Following the argument in [3], we may obtain the change in intensity of the wave Σ_3, knowing its type, which is characterized by the state of the material on the two sides of Σ_3.

As the wave Σ_3 traverses the zone of plastic deformation between Σ_1 and Σ_2, it will be a plastic wave if $I_{(3)} \geq 1$, and will be a relaxation wave if $I_{(3)} < 1$. Calculating the intensity of the tangential stresses $I = \frac{1}{2} s_{ij} s_{ij}$ behind the surface Σ_3, we obtain a condition for the determination of the wave type

$$I_{(3)} = \left(\frac{\omega_3}{\omega_{3\infty}}\right)^2 [3\zeta\sin^2 2\varphi + (1-\zeta)^2], \qquad \omega_{3\infty} = \frac{\sqrt{3}}{\alpha-\beta} \tag{2.7}$$

where $\omega_{3\infty}$ is the intensity of the wave surface Σ as $t \to \infty$.

An analysis of relations (2.7) shows that for ϕ a little differing from zero and for $\omega_3/\omega_{3\infty}$ finite, $I < 1$, i.e., the reflected irrotational wave Σ_3 is a relaxation wave. However for $\phi_1 < \phi < \phi_2$, where ϕ_1 and ϕ_2 are obtained from the condition $I = 1$, Σ_3 may become a plastic wave.

At the point $M_1^{(1)}$ behind the wave surface, the material undergoes elastic stress to the left of the point $M_1^{(1)}$ and plastic to the right. Since the intensity $\omega_3^{(1)}$ of the wave surface Σ_3 changes continuously along Σ_3, the point M_1 may be the source of a weak disturbance which according Huyghens' principle will propagate in the form of an axisymmetric wave surface Σ_5 on which the third derivatives of σ_{ij} and v_i will be discontinuous, so that the surface will separate zones of elastic and plastic behavior behind Σ_5 (Fig. 2).

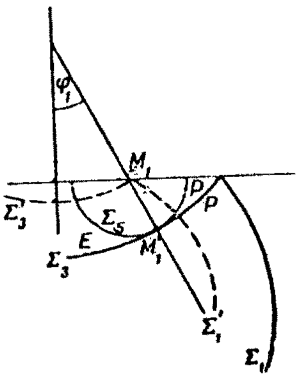

Fig. 2

For $\phi_2 < \phi < \frac{1}{2}\pi$ the reflected irrotational wave Σ_3 again becomes a relaxation wave and at the point M_4 $(\phi = \phi_2)$ on Σ_3 the same phenomenan occurs as does at M_1.

The dependence $F = I(\omega_{3\infty}/\omega_3)^2$ is given in Fig. 3 for different values of Poisson's ratio $\sigma = 0, 0.1, \ldots, 0.5$. From the figure it follows that $\sigma = 0.3$ to 0.5 for $\phi_{10} < \phi < \phi_{20}$ where $\phi_{10} \approx 20°$ and $\phi_{20} \approx 70°$; a spherical reflected wave at finite $\omega_3/\omega_{3\infty} > 1$ is always reflected by a plastic wave. For $0 < \phi < \phi_{10}$ and $\phi_{20} < \phi < \pi/2$ the reflected wave surface Σ_3 at the instant of reflection is a relaxation wave.

The intensity of the relaxation wave Σ_3 varies during its propagation according to Eq. [3]

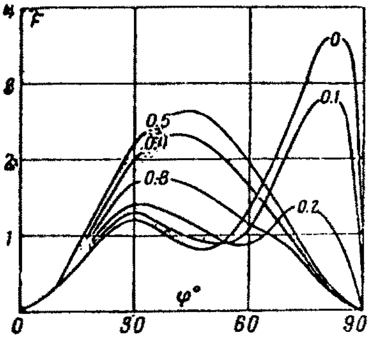

Fig. 3

$$\frac{\delta\omega_3^{(1)}}{\delta t}+\frac{1}{t}\omega_3^{(1)}=[\varepsilon_{ij}^{p}]\,\nu_i^{(3)}\nu_j^{(3)} \tag{2.8}$$

$$[\varepsilon_{ij}^{p}]\,\nu_i^{(3)}\nu_j^{(3)}={}^1/_2\gamma\,({}^2/_3-\sin^2 2\varphi)(\sigma_1-\sigma_2+\sqrt{3})$$

$$\sigma_1-\sigma_2=-(\alpha-\beta)\,\omega_3-\varepsilon(\varphi_1-\varphi_2)-\ldots$$

$$\varepsilon=t-r,\quad r=\sqrt{t^2-4t\cos\varphi+4}$$

The time t_1 of traversal of the wave Σ_3 of the zone of plastic deformation between Σ_1 and Σ_2 is determined by

$$t_{10}=\frac{1}{\cos\varphi}\qquad(0\leqslant\varphi\leqslant\varphi_1) \tag{2.9}$$

$$t_{20}=\frac{4-\varepsilon^2}{2(2\cos\varphi-\varepsilon)}=\frac{1}{\cos\varphi}+\varepsilon\frac{1}{\cos^2\varphi}+\ldots$$

Integrating relations (2.8) under conditions (2.9), we obtain

$$\omega_3^{(1)}=\frac{C_5}{t}+\frac{\gamma(\alpha-\beta)}{\alpha t}\left(\frac{2}{3}-\sin^2 2\varphi\right)\int_{t_{10}}^{t}\left[C_1e^{-g(\alpha-\beta)t}+\frac{\sqrt{3}}{g(\alpha-\beta)^2}-\right.$$

$$\left.-t\left(t-\sqrt{t^2-4t\cos\varphi+4}\right)\left(\varphi_3+\frac{\partial\omega_3}{\partial t}+2\gamma\omega_3-\frac{\omega_3}{t}+\frac{2\sqrt{3}\gamma}{\alpha-\beta}\right)\right]dt \tag{2.10}$$

The constant of integration C_5 is found from the condition $t=t_{10}$, $\omega_3^{(1)}=-\zeta\omega_2$. It should be noted that for $\phi=\frac{1}{2}\arcsin\sqrt{2/3}$ the intensity of the wave Σ_3 changes according to the law of propagation of elastic waves, if $\frac{1}{2}\arcsin\sqrt{2/3}<\phi$. At the instant $t=t_{20}$ the reflected irrotational wave Σ_3 enters the zone of elastic stress behind the surface Σ_2. From the condition of dynamic compatibility (2.1) at the intersection of the wave fronts Σ_2 and Σ_3 it follows that their intersection is regular, i.e. the intersecting waves do not interact.

We study the character of the wave Σ_3 as it enters the elastic zone. From the condition of dynamic compatibility (2.1) we calculate I behind Σ_3 for $t=t_{20}$.

$$I=\left(\frac{\omega_3^{(1)}(t_{20})}{\omega_{3\infty}}\right)^2 \tag{2.11}$$

where

$$\omega_3^{(1)}(t_{20})=-\zeta\omega_3(t_{10})+\varepsilon\,({}^2/_3-\sin^2 2\varphi)\times$$

$$\times\left\{\frac{\omega_3(t_{10})}{2t_{10}}\zeta+\frac{\gamma\sqrt{3}}{2\alpha g(\alpha-\beta)t_{10}}+\frac{C_1\gamma(\alpha-\beta)}{\alpha t_{10}}e^{-g(\alpha-\beta)t_{10}}\right\} \tag{2.12}$$

From relations (2.11) and (2.12) it follows that for $\phi\approx 0$, $\zeta\approx 1$ the reflected irrotational wave Σ_3 enters the elastic zone as a plastic stress wave with intensity $\omega_3^{(1)}(t_{20})$ and propagates further according to the law (2.8).

In the firection $\phi_1<\phi<\phi_2$, the reflected wave Σ_3 is a plastic wave and dies out for small $\omega_3^{(1)}$ according to the law [3]

$$\omega_3^{(1)}(t_2)=\frac{\omega_3^{(1)}(t_{10})}{t}\exp\left\{-\frac{4}{3}\gamma\int_{t_{10}}^{t_2}\left[1-\sqrt{\frac{2}{s_{ij},s_{ij}}}\left(1-\frac{2(s_{ij}\nu_i^{(3)}\nu_j^{(3)})^2}{s_{ij}s_{ij}}\right)\right]dt\right\}$$

$$s_{ij}\nu_i^{(3)}\nu_j^{(3)}=({}^2/_3-\sin^2 2\varphi)(\sigma_1-\sigma_2) \tag{2.13}$$

$$s_{ij}s_{ij}={}^2/_3(\sigma_1-\sigma_2)^2\qquad(t_{10}\leqslant t\leqslant t_{20})$$

Here t_{20} increases as $\phi\to\pi/2$ and the expansion of t_{20} in powers of ε diverges. On entering the elastic zone behind the surface Σ_2 the relaxation wave Σ_3 may be a

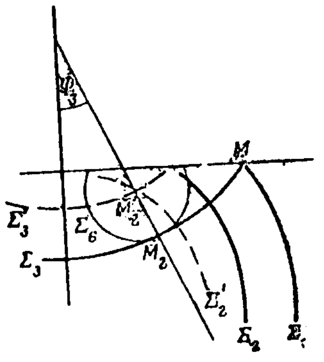

Fig. 4

plastic stress wave ($\phi_2 \leq \phi \leq \phi_3$) or an elastic wave ($\phi_3 < \phi < \pi/2$), depending on the sign of the inequality $\omega_3^{(1)}(t_{20}) \gtrless \omega_{300}$.

The point M_2 on the wave surface Σ_3 is a point of transition from plastic deformation behind Σ_3 to elastic behind Σ_3 at the instant of intersection of the wave surfaces Σ_2 and Σ_3 and is a source of weak disturbance, which will propagate in the form of a toroidal wave surface Σ_6, on which the third derivatives of σ_{ij} and v_i may be discontinuous, so that behind Σ_3 in the region M_0M_2 the medium is plastically deformed, and in the region M_2M, elastically (Fig. 4).

Thus the reflected irrotational wave Σ_3 will propagate in the zone behind Σ_2 as a stress wave for $0 \leq \phi \leq \phi_3$ according to the law (2.8) and as an elastic wave for $\phi_3 < \phi < \pi/2$.

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